Time limit: 45 minutes.
No calculators
Instructions The following questions are meant to be solved by a team of up to four people and require full, well-formulated proofs as answers. Partial credit will be given to solutions that do not completely answer a problem or those with errors.

Introduction:

The game of chess has existed for about 1500 years in various forms, owing in large part to the multitude of different unique games that can be played and the high level of logic required to master the game. A typical chess board is made up of an 8x8 grid with alternating black and white tiles. This team test will focus on chess boards of various sizes as well as the knight piece in order to test your spatial and logical acuity. Before we begin, it may be useful to go over a few definitions and rules. The knight in chess moves in an "L" shape by moving one space in one direction and two spaces in an adjacent direction. A knight's tour then is made by moving a knight through every space on a board, without touching the same space more than once. A knight's circuit is made by forming a knight's tour where the knight ends on the same space it began (meaning that the only space touched more than once is the start and finish space). For example, consider a 3x3 board. On such a board, it would neither be possible to form a knight's tour or circuit, as the center tile does not connect to any of the other eight tiles.

1. Begin with a standard chessboard from which two opposite corners have been removed. Is it possible to cover the board perfectly using dominoes, if each domino must rest on two adjacent squares? Prove your answer.

2. (a) Begin with a standard black/white chessboard. You may repaint entire rows or columns of the board with black or white paint. Is it possible to repaint the board in this way such that it has 1 black square and 63 white squares? Prove your answer.

   (b) Is the same thing possible if you can instead invert the colors of the squares in an entire row or column? Prove your answer.

3. Suppose there is a round-robin chess tournament with n players participating. We can model such a tournament in the following way: Draw a dot for every player. Connect the dots pairwise using arrows, such that the arrow points from the winner of a game to the loser of that game (assume no stalemates).

   (a) Prove that if every player in the tournament finishes with the same number of wins, then n must be odd.

   (b) Prove that if n is odd, there exists a tournament such that every player finishes with the same number of wins.

   (c) Call a player a "King" if, beginning from their dot, we can reach every other dot by following no more than two arrows. Prove that every such tournament has a king.

4. Prove that there is a knight's **tour** on a 3x4 board.

5. (a) Prove that there is no knight's **circuit** on a 3x5 board.

   (b) Prove that there is no knight's **circuit** on an mxn board where m and n are both odd

6. Given that there are knight's **circuits** on a 3x10 and on a 3x12 board, use Problem 1 to prove that there is a knight's **tour** on any 3xn board where $n > 8$ is even.

7. Given that there are knight's **tours** on 3x7, 3x8, and 3x9 boards, prove that there is a knight's **tour** on any 3xn board where $n \geq 7$.

8. (a) Prove that there are no knight's **tours** on a 4x4 board.

   (b) Prove that there are no knight's **circuits** on a 4xn board where n is any whole number.